JOURNAL OF APPROXIMATION THEORY 51, 160–162 (1987)

The Failure of Bernstein's Theorem for Polynomials on $C(K)$ Spaces

ANDREW TONGE

Department of Mathematics and Statistics, Brunel University, Kingston Lane, Uxbridge, Middlesex UB8 3PH, England

Communicated by P. L. Butzer

Received May 7, 1985; revised January 28, 1986

Bernstein's theorem asserts that if $p: \mathbb{C} \to \mathbb{C}$ is a polynomial of degree m, then its derivative p' satisfies the inequality $\|p'\|_\infty \leqslant m \|p\|_\infty$, where the symbol $\|\ \|_\infty$ denotes the supremum norm taken over the unit disc. Harris [2] proved an analogous inequality for the Fréchet derivative of polynomials on Hilbert space. In his commentary to problem 73 in the Scottish Book (R. D. Mauldin, Ed., Birkhäuser, Boston, pp. 144–145, 1981), he asked whether there is a similar result for polynomials on $C(K)$ spaces. The purpose of this note is to give a negative answer, even for polynomials of degree 2. © 1987 Academic Press, Inc.

THE EXAMPLE

We recall the definition of a polynomial on a Banach space. If $L: E \times \cdots \times E \to \mathbb{C}$ is a continuous symmetric m-linear form on the Banach space E, we define the map $\hat{L}: E \to \mathbb{C}$ by $\hat{L}(x) = L(x, ..., x)$. A map $p: E \to \mathbb{C}$ is

(a) a homogeneous polynomial of degree 0 if p is constant,

(b) a homogeneous polynomial of degree $m \geqslant 1$ if $p = \hat{L}$ for some continuous symmetric m-linear form L on E, and

(c) a polynomial of degree m if $p = p_0 + \cdots + p_m$, where p_i is a homogeneous polynomial of degree i $(0 \leqslant i \leqslant m)$ and $p_m \not\equiv 0$.

If $p: E \to \mathbb{C}$ is a polynomial with Fréchet derivative Dp, we define

$$\|p\|_\infty = \sup\{|p(x)|: \|x\| \leqslant 1\}$$

and

$$\|Dp\|_\infty = \sup\{\|Dp(x)\|: \|x\| \leqslant 1\} = \sup\{|Dp(x)(y)|: \|x\| \leqslant 1, \|y\| \leqslant 1\}.$$

0021-9045/87 $3.00

We use the standard notation l_∞^n for the space $\mathbb{C}^n$ equipped with the norm $\|x\|_\infty = \|(x_1,\ldots, x_n)\|_\infty = \max\{|x_1|,\ldots, |x_n|\}$.

In [2], Harris proved that if $p: E \to \mathbb{C}$ is a polynomial of degree m then $\|Dp\|_\infty \leqslant m\|p\|_\infty$, provided that E is a Hilbert space or that $E = l_\infty^2$. This prompted him to ask whether the same inequality holds whenever $E = C(K)$, the Banach space of continous functions on the compact Hausdorff space K, under the usual uniform norm. We shall give an example of a homogeneous polynomial of degree 2 on the space $E = l_\infty^3$ for which the proposed inequality fails.

Define a symmetric bilinear form $L: l_\infty^3 \times l_\infty^3 \to \mathbb{C}$ by

$$L(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_1 y_2 - x_2 y_1 - x_2 y_3 - x_3 y_2 - x_3 y_1 - x_1 y_3,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The corresponding polynomial, which has already been useful in the investigation of von Neumann's inequality [7], is given by

$$\hat{L}(x) = x_1^2 + x_2^2 + x_3^2 - 2x_1 x_2 - 2x_2 x_3 - 2x_3 x_1.$$

In [7] Kaijser and Varopoulos used elementary calculus to show that $\|\hat{L}\|_\infty = 5$. (The norm is attained when $x = (1, 1, -1)$.) Consequently we need to show that $\|D\hat{L}\|_\infty > 10$.

However, if we set $x = (1, w, w^2)$ and $y = (1, w^2, w)$, where $w = \exp(2\pi i/3)$, we obtain $D\hat{L}(x)(y) = 2L(x, y) = 12$. It follows that $\|D\hat{L}\|_\infty \geqslant 12$.

Comments and Open Problems

It has been shown that if p is a homogeneous polynomial of degree 2 on a $C(K)$ space then $\|Dp\|_\infty \leqslant \sqrt{(27/4)}\, \|p\|_\infty$. A proof may be found (implicitly) in [2] or [6]. Computer experiments strongly suggest that this constant is best possible, but we know of no proof.

It would be interesting to classify the Banach spaces for which Bernstein's theorem does hold.

ADDENDUM

I am grateful to the referee for pointing out that the example gives a counter-example to a theorem of Michal [4]. On page 125 of his paper, he claims that "in an arbitrary complex Banach space the analogue of Bernstein's inequality always holds." Further counter-examples exist. The reader may like to consult Example 1.8 in the book of Dineen [1] or refer to the paper of Sarantopoulos [5].

REFERENCES

1. S. DINEEN, Complex Analysis in Locally Convex Spaces, Ann. of Math. Stud. Vol. 57, North-Holland, Amsterdam, 1981.
2. L. A. HARRIS, Bounds on the derivatives of holomorphic functions of vectors *in* "Colloque d'analyse," (L. Nachbin, Ed.) Rio de Janeiro, 1972, Herman, Paris, *Acta Sci. et Ind.* **1367** (1975), 145–163.
3. R. D. MAULDIN (Ed.), "The Scottish Book (Mathematics from the Scottish Café)," Birkhäuser, Boston, 1981.
4. A. D. MICHAL, On bounds of polynomials in hyperspheres and Fréchet-Michal derivatives in real and complex normed linear spaces, *Math. Mag.* **27** (1953–4), 119–126.
5. I. SARANTOPOULOS, Estimates for polynomial norms on $L_p(\mu)$ spaces, *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 263–271.
6. A. M. TONGE, Polarization and the two dimensional Grothendieck inequality, *Math. Proc. Cambridge Philos. Soc.* **95** (1984), 313–318.
7. N. TH. VAROPOULOS, On an inequality of von Neumann and an application of the metric theory of tensor products to operator theory, *J. Funct. Anal.* **16** (1974), 83–100.